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1985 J. Phys. A: Math. Gen. 18 L735

(http://iopscience.iop.org/0305-4470/18/12/009)

J. Phys. A: Math. Gen. 18 (1985) L735-L738. Printed in Great Britain

LETTER TO THE EDITOR

Destruction of the devil's staircase

G Forgacs† and Zhang Yi-Cheng

Physics Department, Brookhaven National Laboratory, Upton, Long Island, NY 11973, USA

Received 3 June 1985

Abstract. The one-dimensional Ising model with oscillating interaction is considered in the presence of a magnetic field. The devil's staircase found for convex interaction is destroyed and the system crosses over to a harmless staircase.

It has been shown (Bak and Bruinsma 1982) that the ground states of the one-dimensional Ising model with infinite range convex antiferromagnetic interaction form a devil's staircase (Aubry 1980) as a function of the external magnetic field. If the interaction is of arbitrary finite range, but still convex, the devil's staircase crosses over to a harmless staircase (Shinjo and Sasada 1985). On the other hand, in the one-dimensional Ising model with ferromagnetic nearest-neighbour and antiferromagnetic next-nearest-neighbour interactions the devil's staircase also decays to a harmless one (Villain and Gordon 1980). In the latter model the interaction is non-convex and of short range.

At this point one may ask the question what is responsible for the destruction of the devil's staircase. Is it the finite range of the interaction or its non-convexity or both.

In the present letter we study the case of an infinite range, oscillating interaction. We show that there cannot be a devil's staircase even in this case. By this we arive at the conclusion that if any of the conditions put on the interaction in the Bak-Bruinsma (BB) model (or the equivalent Hubbard model, Hubbard 1978) is not satisfied, the devil's staircase disappears.

Our model is defined by the Hamiltonian

$$\mathscr{H} = -H \sum_i S_i + \tfrac{1}{2} \sum_{ij} J(|i-j|)(S_i+1)(S_j+1) \tag{1}$$

where S_i are Ising spins (± 1) on a one-dimensional lattice. The summation in the second pair extends over all pairs with $i \neq j$. The interaction J is chosen as

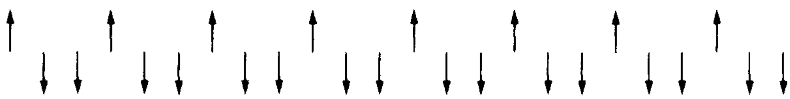
$$J(|i-j|) = (-1)^{P-1}/|i-j|\alpha. \tag{2}$$

$J(|i-j|)$ gives the interaction between up spins S_i and S_j which are Pth neighbours ($P = 1, 2, \ldots$) and α is typically a positive integer of order one. Just as in the BB case, for $H < 0$ the unique ground state of (1) is $S_k = -1$ for all k, whereas for $H > 2\sum_{i>0} J(i)$ (this sum in our case is convergent for any α) the ground state is $S_k = +1$. Following Hubbard (Hubbard 1978) we can construct the ground state for any rational fraction

† Present address: CEA, SPSRM, Orme des Merisiers, Gif-sur-Yvette Cedex 91191, France.

$q = N_+/N = m/n$ with fixed N_+. Here N_+ and N correspond to the number of up spins and total number of lattice sites respectively.

It turns out that the ground states in the present case fall into two groups depending on whether m is even or odd. In the odd case the structure follows exactly that of Hubbard (Hubbard 1978, Bak and Bruinsma 1982). For example, for $q = \frac{1}{3}$ one can show that by shifting any number of up spins in figure 1(a) the energy will be increased. For $q = \frac{2}{3}$ the ground state is given in figure 1(b). The general rule for constructing the ground state in the even case follows from this figure. One first determines the locations of the up spins in the corresponding Hubbard case (J given by (2) with $P = 1$ for any pair; Hubbard 1978) within the period n. (For $q = m/n$, there are m up spins in the period n; the lattice constant is unity.) Then one reverses the order of the up spins in every second period relative to the Hubbard case, as shown for $q = \frac{2}{3}$ in figure 1(b).


(a)


(b)

Figure 1. The ground state for (a) $q = \frac{1}{3}$ and (b) $q = \frac{2}{3}$.

If J is convex, Bak and Bruinsma proved that for any q there is a finite interval in H in which the state with this q is stable. This led to the devil's staircase. Following Bak and Bruinsma we can also study the stability interval of our states by comparing their energy with that of a state obtained by flipping one spin (either up or down). By flipping a single spin the system will rearrange itself by creating solitons in order to reduce the energy. In the BB case such a spin flip always costs energy, because the created solitons repel each other. This leads finally to a finite stability interval in H. In the present case it turns out that for both even and odd m the solitons have attractive interaction energy and therefore stay together. One can show that for a given q the number of solitons, created by flipping a single spin, is n and the number of up spins l_s within one soliton is given by the solution of

$$nl_s = 2m\,k \pm 1 \tag{3}$$

with the smallest integer k. For even m, k gives the number of periods which are covered by the n solitons. For odd m, the same number is given by $2k$. (Remember that for even m the period is $2n$.) The $\pm$ signs in (3) correspond to flipping one down spin up (+) or one up spin down (−). For example, for $q = \frac{1}{3}$, $n = 3$, $m = 1$ and from (3) in the case of flipping a down spin up $k = 1$, $l_s = 1$. For $q = 2/3$, $k = 2$, $l_s = 3$. Using (3) one can easily calculate the width L of a soliton. One gets that $L = 2k$ in both the even and odd cases. The above considerations are summarised in figure 2 for $q = \frac{1}{3}$ and $\frac{2}{3}$, where the corresponding soliton structures are shown for the + sign in (3).

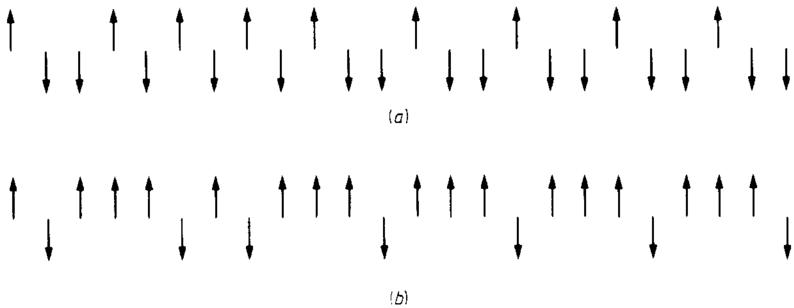


Figure 2. Soliton structure for (a) $q=\frac{1}{3}$ and (b) $q=\frac{2}{3}$. Note that the number of up spins is increased by one with respect to figure 1.

Since the interaction between solitons is attractive, the more solitons we have (the larger n is) the more they will reduce the energy of the 'excited' (flipped) state. If for a given q we denote by E_q and E'_q the energies of the ground state and the 'excited' state respectively, then one can write

$$\Delta E_q = E'_q - E_q = A_q - nB_q \tag{4}$$

where A_q is the energy required to flip a spin without considering the interaction of the solitons and nB_q is the total soliton interaction energy. In the BB case, since the solitons repel each other, one can separate them infinitely apart and by this one can ignore their interaction. A_q has been shown (Bak and Bruinsma 1982) to be a finite positive number, not to be proportional to n. In the present case one cannot separate the solitons; they will stick together and $B_q > 0$. It is then evident from (4) that for q's with sufficiently large n the second term may overcome the first one. This implies the stability interval of the found ground state is zero. If a given state is unstable with respect to single spin flips it is favourable to flip more spins to saturate the number of solitons to creat a soliton lattice. This soliton lattice, however, is nothing but another commensurate state with a larger q (if down spins are flipped up). In order to find out whether this new state is stable or not, one can repeat the above procedure by flipping spins until ΔE_q in (4) is positive.

Since it is those q values with large n which in the BB case give the fine structure of the devil's staircase, this fine structure in the present case disappears.

We were not able to calculate analytically A_q and B_q and, therefore, could not determine at which n the above discussed stability sets in. It is likely, however, that states with small n can survive (one does not expect all the commensurate states to become unstable) and so the devil's staircase crosses over to a harmless staircase.

We wish to thank Per Bak and R Bruinsma for useful discussions. One of us (GF) thanks the Brookhaven National Laboratory for the hospitality during his stay when this work was performed.

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